Nonuniqueness of Representations of Wave Equations in Lorentzian Space-Times

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Abstract

This brief note wants to bring to attention that the formulation of physically reasonable initial-boundary value problems for wave equations in Lorentzian space-times is not unique, i.e., that there are inequivalent such formulations that lead to a different outcome of the stability discussion of the solutions. For demonstration, the paper uses the case of the wave equation on the right Rindler wedge in 2-dimensional Minkowski space. The used methods can be generalized to wave equations on stationary globally hyperbolic space-times with horizons in higher dimensions, such as Schwarzschild and Kerr space-times.

1 Introduction

The stability discussion of solutions of Einstein's field equations usually lead on wave equations in Lorentzian space-times, describing perturbations of the metric, together with physically boundary conditions. In a second step, the definition of the precise class of the considered solutions is specified, i.e., a data space is chosen for the solutions of the equations that leads on a well-posed initial-boundary value problem. The latter provides the basis for a meaningful discussion of the stability of the solutions, i.e., the existence or non-existence of exponentially growing solutions.

This brief note wants to bring to attention that this process is not unique, i.e., that there are inequivalent ways of formulating initial-boundary value problems for such wave equations

and at the same time that the choice of the formulation can affect the outcome of the stability discussion. One such example is given by the treated relatively simple case of the wave equation on the right Rindler wedge in 2-dimensional Minkowski space.

2 Restriction of the Wave Equation to the Right Wedge of 2-dimensional Minkowski Space-Time

In the following, we consider the solutions of the wave equation on the right wedge,

$$R := \{(x^0, x^1) \in \mathbb{R}^2 : x^1 > |x^0|\} ,$$

of 2-dimensional Minkowski space-time $(\mathbb{R}^2, g)$, where

$$g = (\mathbb{R}^2, dx^0 \otimes dx^0 - dx^1 \otimes dx^1) ,$$

and $(x^0, x^1) : \mathbb{R}^2 \to \mathbb{R}^2$ denotes an inertial coordinate system. [1] For the coordinatization of R, we use well-known "Rindler coordinates" $(\tau, \xi) : R \to \mathbb{R}^2$ given by

$$\tau(x^0, x^1) := \frac{1}{2} \ln\left(\frac{x^1 + x^0}{x^1 - x^0}\right) , \ \xi(x^0, x^1) := \frac{1}{2K} \ln\left(K^2[(x^1)^2 - (x^0)^2]\right) \tag{1}$$

for all $(x^0, x^1) \in R$. Here $K > 0$ is a constant having the dimension $1/$length. The inverse transformation to (1) is given by

$$(\tau, \xi)^{-1}(\tau, \xi) = \frac{1}{K} \left(e^{K\xi} \sinh(\tau), e^{K\xi} \cosh(\tau)\right)$$

for every $(\tau, \xi) \in \mathbb{R}^2$, and the restriction $g|_R$ of g to R is given by

$$g|_R = e^{2K\xi} \left(\frac{1}{K^2}\, d\tau \otimes d\tau - d\xi \otimes d\xi\right) .$$

Solutions $u \in C^2(R, \mathbb{C})$ of the wave equation on R satisfy

$$\Box u = K^2 e^{-2K\xi} \partial_\tau^2 u - e^{-2K\xi} \partial_\xi^2 u = 0 , \tag{2}$$

or equivalently

$$\partial_\tau^2 u = -\left(-\frac{1}{K^2}\, \partial_\xi^2\right) u , \tag{3}$$

[1] If not otherwise indicated, the symbols x^0, x^1, τ, ξ denote coordinate projections whose domains will be obvious from the context. In addition, we assume the composition of maps, which includes addition, multiplication and so forth, always to be maximally defined. For instance, the sum of two complex-valued maps is defined on the intersection of their domains.

$$e^{-K\xi}\partial_\tau^2\left(e^{-K\xi}u\right) = -e^{-K\xi}\left[e^{-K\xi}\left(-\frac{1}{K^2}\,\partial_\xi^2\right)e^{K\xi}\right]\left(e^{-K\xi}u\right)\ . \tag{4}$$

Of course, many other forms of (2) are possible. In the following, we will consider only (3) and (4).

As said, both equations are equivalent on the level of C^2-solutions. For arbitrarily given data $g_1 \in C^2(\mathbb{R},\mathbb{R})$ and $g_2 \in C^1(\mathbb{R},\mathbb{R})$, there is a unique solution $u \in C^2(\mathbb{R}^2,\mathbb{R})$ to these equations such that

$$u(0,\xi) = g_1(\xi)\ ,\ \frac{\partial u}{\partial \tau}(0,\xi) = g_2(\xi)$$

for every $\xi \in \mathbb{R}$. Moreover, this solution is given by

$$u(\tau,\xi) = \frac{1}{2}\left[g_1\left(\xi + \frac{\tau}{K}\right) + g_1\left(\xi - \frac{\tau}{K}\right) + K\int_{\xi-\frac{\tau}{K}}^{\xi+\frac{\tau}{K}} g_2(s)\,ds\right] \tag{5}$$

for all $(\tau,\xi) \in \mathbb{R}^2$, and

$$\begin{aligned} u(x^0,x^1) = \frac{1}{2}\Bigg\{ & g_1\left(\frac{1}{K}\ln[K(x^1+x^0)]\right) + g_1\left(\frac{1}{K}\ln[K(x^1-x^0)]\right) \\ & + K\int_{\frac{1}{K}\ln[K(x^1-x^0)]}^{\frac{1}{K}\ln[K(x^1+x^0)]} g_2(s)\,ds\Bigg\} \end{aligned} \tag{6}$$

for all $(x^0,x^1) \in R$.

On the other hand, (3) results from (2) by solution for the highest time derivative. In this, (2) is divided by the unbounded function $K^2e^{-2K\xi}$, and later (3) will be treated analogous to the wave equation on 2-dimensional Minkowski space. In this step geometrical information is lost. $(R, g|_R)$ and Minkowski space are both globally hyperbolic, but $(R, g|_R)$ is geodesically incomplete, i.e., there are maximal geodesics whose domains are proper subsets of $\mathbb{R}$, whereas Minkowski space is geodesically complete. It needs to be stressed that such loss of geometrical information is not particular to wave equations on 2-dimensional Lorentzian space-times, but also happens in higher dimensions. In addition, later (3) will be treated using methods from operator theory, where an unbounded function corresponds to an unbounded (or "discontinuous") operator. Therefore, also from an operator theory perspective, the "division" of (2) by $K^2e^{-2K\xi}$ needs consideration.

We are going to see that natural functional analytic treatments of (3), (4) lead to inequivalent well-posed initial value formulations for (2). In this connection, it needs to be taken into account that (2) admits C^2-solutions of stronger than exponential growth in space and time, for instance,

$$u(\tau,\xi) = \exp\left\{\frac{\alpha}{K}\exp\left[K\left(\xi \pm \frac{\tau}{K}\right)\right]\right\}$$

where $(\tau, \xi) \in \mathbb{R}^2$ and $\alpha \in \mathbb{C}$. Therefore, the class of C^2-solutions does not provide a meaningful framework for discussions of stability, and hence some form of functional analytic treatment is necessary, at least a restriction of the space of admissible data.

3 A Common Functional Analytic Representation of (3)

(3) results from (2) by solution for the highest time derivative. In this, the whole equation is divided by the unbounded function $K^2 e^{-2K\xi}$. In the next step, (3) is represented as a member of the class of abstract evolution equations, see e.g., [6],

$$u''(t) = -A\,u(t)\ , \tag{7}$$

$t \in \mathbb{R}$, where $A : D(A) \to X$ is some densely-defined, linear, positive self-adjoint operator in some non-trivial complex Hilbert space $(X, \langle | \rangle)$.

For every equation from this class and for any g_1, g_2 from the domain $D(A)$ of the corresponding operator A, there is a uniquely determined twice continuously differentiable map $u : \mathbb{R} \to X$ assuming values in $D(A)$ and satisfying (7) for all $t \in \mathbb{R}$ as well as

$$u(0) = g_1\ ,\ u'(0) = g_2\ .$$

It is important to note that, mainly as a consequence of the self-adjointness of A, this approach leads automatically to a conserved energy.[1] For this u, the corresponding canonical energy function $E_u : \mathbb{R} \to \mathbb{R}$, defined by

$$E_u(t) := \frac{1}{2}\left(\ \langle u'(t)|u'(t)\rangle + \langle u(t)|Au(t)\rangle\ \right)$$

for all $t \in \mathbb{R}$, is constant.

Finally, if $B : D(B) \to X$ is some square root of A, i.e., some densely-defined, linear, self-adjoint operator commuting with A which satisfies

$$B^2 = A\ ,$$

for example, $B = A^{1/2}$, then this u is given by

$$u(t) = \cos(tB)g_1 + \frac{\sin(tB)}{B}g_2 \tag{8}$$

for all $t \in \mathbb{R}$ where $\cos(tB), \sin(tB)/B$ denote the bounded linear operators that are associated by the functional calculus for B to the restrictions of $\cos, \sin/\mathrm{id}_{\mathbb{R}}$ to the spectrum

[1] This energy corresponds to the canonical energy of the classical field u, described by (3), see e.g., [13].

of B.

We note that

$$\cos(tB)\ ,\ \frac{\sin(tB)}{B}\ ,$$

for every $t \in \mathbb{R}$, are bounded linear operators. This leads to "generalized" solutions of (7) for arbitrary data from X. For such generalized solutions, the corresponding "energy" is ill-defined. On the other hand, analogously to the Schroedinger equation of quantum theory, where such generalized solutions are of course physical, not only elements from the domain of the Hamilton operator are admissible quantum states and are subject to time evolution [1], it does not appear reasonable to discard such generalized solutions from consideration. Analogous to Schroedinger theory, where the Schroedinger equation is merely a "label" for the generalized solutions given by the corresponding unitary one-parameter group, (7) might be considered as a "label" for (8) and the latter being the truly relevant object for applications.

Also, we note that (8) implies that

$$\|u(t)\| \leq \|g_1\| + |t|\|g_2\|$$

for $t \in \mathbb{R}$ and hence *that the solutions of (7) are stable in the sense that there are no exponentially growing solutions.* [2]

In our special case, (3), $X = L^2_{\mathbb{C}}(\mathbb{R})$, A is closure of the densely-defined, linear, positive symmetric and essentially self-adjoint operator in $L^2_{\mathbb{C}}(\mathbb{R})$

$$A_0 := \begin{pmatrix} C_0^\infty(\mathbb{R},\mathbb{C}) \to L^2_{\mathbb{C}}(\mathbb{R}) \\ f \mapsto -\frac{1}{K^2} f'' \end{pmatrix}\ .$$

Application of (8) gives, see Theorem 6.2 in the Appendix, a representation of the solutions of (7) given by

$$\begin{aligned} u(\tau) &= \cos(\tau\bar{p}_\xi) g_1 + \frac{\sin(\tau\bar{p}_\xi)}{\bar{p}_\xi}\, g_2 \qquad (9) \\ &= \frac{1}{2}\left[g_1 \circ \left(\mathrm{id}_{\mathbb{R}} + \frac{\tau}{K}\right) + g_1 \circ \left(\mathrm{id}_{\mathbb{R}} - \frac{\tau}{K}\right) + K\,\mathrm{sgn}(\tau)\left(\chi_{[-|\tau|/K,|\tau|/K]} * g_2\right)\right] \end{aligned}$$

for every $g_1, g_2 \in D(A)$. Here $\circ$ denotes composition, $\mathrm{id}_{\mathbb{R}}$ the identical function on $\mathbb{R}$,

$$\mathrm{sgn} := \chi_{(0,\infty)} - \chi_{(-\infty,0)}\ ,$$

[1] See e.g., Section 2.1 in [6].

[2] Note that the results of this section have generalizations to semibounded A, e.g., see Corollary 2.2.2 in [6]. In particular, differently to positive A, for non-positive A, there are exponentially growing solutions to (7).

$*$ denotes the usual convolution product, and $\bar{p}_\xi$ is the closure of the densely-defined, linear, symmetric and essentially self-adjoint operator p_ξ in $L^2_{\mathbb{C}}(\mathbb{R})$ given by

$$\begin{pmatrix} C_0^\infty(\mathbb{R},\mathbb{C}) \to L^2_{\mathbb{C}}(\mathbb{R}) \\ f \mapsto \frac{i}{K} f' \end{pmatrix} \ .$$

As a side remark, $\bar{p}_\xi$ is a square root of A, i.e., $\bar{p}_\xi^2 = A$, that commutes with A, but $\bar{p}_\xi$ is different from the positive square root, $A^{1/2}$, of A.

We note that, as had to be expected, essentially (9) is just a natural generalization of (5) to the elements of the domain $D(A)$ of A. Since also

$$\begin{aligned} &\cos(\tau \bar{p}_\xi) g_1 + \frac{\sin(\tau \bar{p}_\xi)}{\bar{p}_\xi}\, g_2 \\ &= \frac{1}{2}\left[g_1 \circ \left(\mathrm{id}_{\mathbb{R}} + \frac{\tau}{K} \right) + g_1 \circ \left(\mathrm{id}_{\mathbb{R}} - \frac{\tau}{K} \right) + K \,\mathrm{sgn}(\tau) \left(\chi_{[-|\tau|/K, |\tau|/K]} * g_2 \right) \right] \end{aligned}$$

for every $g_1, g_2 \in L^2_{\mathbb{C}}(\mathbb{R})$, see Theorem 6.2 in the Appendix, also the generalized solutions of (7) in our case are given by a natural generalization of (5) to elements of $L^2_{\mathbb{C}}(\mathbb{R})$.

The slight disadvantage of this common functional analytic representation of (3) comes from the fact that the data for (7) are from $L^2_{\mathbb{C}}(\mathbb{R})$ and hence vanish in the mean for $\xi \to -\infty$, i.e., roughly speaking, *vanish in the bifurcation point* $(0,0)$ *of the boundary (horizon)* H *of* R,

$$H = \{(x^0, x^1) \in \mathbb{R}^2 : x^1 = |x^0|\} \ .$$

Indeed, taking into account an embedding of the Rindler wedge into 2-dimensional Minkowski space such behavior appears not natural. On the other hand, this behavior of the solutions is unsurprising since (3) is treated analogous to the wave equation on 2-dimensional Minkowski space, where data vanish in the mean at spatial infinity. Also, *the choice of* $L^2_{\mathbb{C}}(\mathbb{R})$ *as data space is related to the pursued self-adjointness of the operator* A. On the one hand, the self-adjointness of A enables the application spectral theorems for self-adjoint operators which allow the proof of well-posedness of the initial value problem for the evolution equation and also lead on a representation of its solutions. Such theorems are generally not available for non-self-adjoint operators. Also, along with semiboundedness, the self-adjointness of A leads to the existence of a conserved energy.

This disadvantage of the previous method in connection with wave equations on globally-hyperbolic Lorentzian space-times with horizons has been noticed before, among others by [11, 4, 5]. On the other hand, it needs to be stressed that this disadvantage comes into play only if it is known that the corresponding space-times are embedded in a larger

space-time. For instance, the Rindler wedge can be embedded into 2-dimensional Minkowski space. Such embedding information is intrinsically "non-local."

In addition, in the physics literature, the solution of wave equations for the highest time derivative of the unknown along with a subsequent use of the above operator theoretic approach is used in most stability discussions, often implicitly without full realization by the authors, for e.g., see [9]. Such use is indicated, whenever a stability discussion leads to the finding of unstable eigenvalues/spectra or "quasinormal frequencies." For an interpretation of the latter in terms of resonances of self-adjoint operators, see [3]. *Apparently, the only rigorous framework for such discussion is provided by the spectral theory of operators*.

4 A Functional Analytic Representation of (4)

In this connection, we note that the restriction $\mu|_R$ of the volume 2-form $\mu = dx^0 \wedge dx^1$ on $(\mathbb{R}^2, g)$ to R is given by

$$\mu|_R = \frac{e^{2K\xi}}{K}\, d\tau \wedge d\xi \ .$$

Hence $\mu|_R$ induces on surfaces of constant τ the 1-form ("measure")

$$\frac{e^{2K\xi}}{K}\, d\xi \ ,$$

where here and in the following ξ is also used as abbreviation for $\mathrm{id}_{\mathbb{R}}$. Therefore, we choose

$$L^2_{\mathbb{C}}(\mathbb{R}, e^{2K\xi})$$

as data space for our representation of (4). Further, we note that the map V defined by

$$Vf := e^{-K\xi} \cdot f$$

for every $f \in L^2_{\mathbb{C}}(\mathbb{R})$ defines a Hilbert space isomorphism

$$V : L^2_{\mathbb{C}}(\mathbb{R}) \to L^2_{\mathbb{C}}(\mathbb{R}, e^{2K\xi})$$

with inverse

$$V^{-1} = \begin{pmatrix} L^2_{\mathbb{C}}(\mathbb{R}, e^{2K\xi}) \to L^2_{\mathbb{C}}(\mathbb{R}) \\ f \mapsto e^{K\xi} f \end{pmatrix} \ .$$

Employing the previous notation, the expression enclosed by square brackets in (4),

$$e^{-K\xi} \left(-\frac{1}{K^2}\, \partial_\xi^2 \right) e^{K\xi} \ ,$$

is represented by

$$VAV^{-1} \ .$$

The latter operator is densely-defined, linear and positive self-adjoint. The remaining factors $e^{-K\xi}$ in (4) are represented by the corresponding maximal multiplication operator $T_{e^{-K\xi}}$ in $L^2_{\mathbb{C}}(\mathbb{R}, e^{2K\xi})$. We note that $T_{e^{-K\xi}}$ is densely-defined, linear, self-adjoint and bijective. Also, $T_{e^{-K\xi}}$ leaves $C_0^\infty(\mathbb{R}, \mathbb{C})$ invariant. In this way, we arrive at the following functional analytic interpretation of (4)

$$T_{e^{-K\xi}} \left(T_{e^{-K\xi}} u\right)'' = -T_{e^{-K\xi}} V A V^{-1} \left(T_{e^{-K\xi}} u\right) \ , \tag{10}$$

where primes denote derivatives of paths in $L^2_{\mathbb{C}}(\mathbb{R}, e^{2K\xi})$.

Since $T_{e^{-K\xi}}$ is bijective, the latter equation is satisfied if and only if

$$\left(T_{e^{-K\xi}} u\right)'' = -V A V^{-1} \left(T_{e^{-K\xi}} u\right) \ .$$

The latter equation is of type (7). From the results from the previous section as well as the invariance of $C_0^\infty(\mathbb{R}, \mathbb{C})$ under

$$\cos(tB) \ , \ \frac{\sin(tB)}{B} \ ,$$

follows for every $g_1 \in C_0^\infty(\mathbb{R}, \mathbb{C})$, $g_2 \in C_0^\infty(\mathbb{R}, \mathbb{C})$, that there is a unique solution to (10) satisfying

$$u(0) = g_1 \ , \ u'(0) = g_2$$

and that this solution is given by

$$\begin{aligned} u(\tau) &= T_{e^{K\xi}} \cos(\tau V \bar{p}_\xi V^{-1}) T_{e^{-K\xi}} g_1 + T_{e^{K\xi}} \, \frac{\sin(\tau V \bar{p}_\xi V^{-1})}{V \bar{p}_\xi V^{-1}} \, T_{e^{-K\xi}} g_2 \qquad (11) \\ &= T_{e^{K\xi}} V \cos(\tau \bar{p}_\xi) V^{-1} T_{e^{-K\xi}} g_1 + T_{e^{K\xi}} V \, \frac{\sin(\tau \bar{p}_\xi)}{\bar{p}_\xi} V^{-1} \, T_{e^{-K\xi}} g_2 \\ &= \cos(\tau \bar{p}_\xi) g_1 + \frac{\sin(\tau \bar{p}_\xi)}{\bar{p}_\xi} \, g_2 \\ &= \frac{1}{2} \left[g_1 \circ \left(\mathrm{id}_{\mathbb{R}} + \frac{\tau}{K} \right) + g_1 \circ \left(\mathrm{id}_{\mathbb{R}} - \frac{\tau}{K} \right) + K \, \mathrm{sgn}(\tau) \left(\chi_{[-|\tau|/K, |\tau|/K]} * g_2 \right) \right] \end{aligned}$$

for every $t \in \mathbb{R}$, where $T_{e^{K\xi}}$ denotes the maximal multiplication operator in $L^2_{\mathbb{C}}(\mathbb{R}, e^{2K\xi})$ corresponding to $e^{K\xi}$. Further, for $\tau \in \mathbb{R}$, by

$$f \circ \left(\mathrm{id}_{\mathbb{R}} \pm \frac{\tau}{K} \right) \ , \ \chi_{[-|\tau|/K, |\tau|/K]} * f$$

for every $f \in L^2_{\mathbb{C}}(\mathbb{R}, e^{2K\xi})$, there are defined bounded linear operators on $L^2_{\mathbb{C}}(\mathbb{R}, e^{2K\xi})$, see Theorems 6.4, 6.5 in the Appendix.

As had to be expected, also (10) turns out as "label" for the natural generalization of (5) to the elements $L^2_{\mathbb{C}}(\mathbb{R}, e^{2K\xi})$.

Still the functional analytic representations of (3) and (4) are different, since the field u, as a classical field, is observable. Also the field corresponding to (4) does not necessarily vanish in the bifurcation point $(0,0)$ of the horizon. For instance, the restriction of the generalized solution of (4) to

$$W_L := \left\{ (\tau, \xi) \in \mathbb{R}^2 : \xi < \frac{|\tau|}{K} \right\}$$

corresponding to data

$$u(0, \cdot) = e^{-\alpha\xi} \cdot \chi_{(-\infty,0)} \ , \ u'(0, \cdot) = 0 \ ,$$

where $\alpha < K$, is given by

$$u(\tau, \xi) = e^{-\alpha\xi} \cosh\left(\frac{\alpha\tau}{K}\right) \tag{12}$$

for every $(\tau, \xi) \in W_L$. The latter leads to

$$u(x^0, x^1) = \frac{1}{2} \left\{ [K(x^1 + x^0)]^{-\frac{\alpha}{K}} + [K(x^1 - x^0)]^{-\frac{\alpha}{K}} \right\}$$

for every $(x^0, x^1) \in \mathbb{R}^2$ satisfying

$$|x^0| < x^1 < |x^0| + \frac{1}{K} \ .$$

In addition, *the weighted L^2-norm that is corresponding to (12) is exponentially increasing in τ*. Therefore, the solutions to (10) might be considered unstable, *on the other hand with respect to inertial coordinate system (x^0, x^1), no exponential growth is visible*. Of course, such outcome also raises the question of coordinate dependence of the results.

5 Discussion

This brief points out that the formulation of physically reasonable initial-boundary value problems for wave equations in Lorentzian space-times is not unique, i.e., that there are inequivalent such formulations that lead to a different outcome of the stability discussion of the solutions.

For the example of the wave equation on the right Rindler wedge in 2-dimensional Minkowski space, this note gives 2 inequivalent formulations of a well-posed initial-boundary

value problem, leading to different outcomes of the stability discussion of the solutions. Their construction suggest the existence of many more of such formulations. All what needs to be found is a weight ρ such that the family of linear operators on $C_0^\infty(\mathbb{R}, \mathbb{C})$, defined by

$$\frac{1}{2}\left[\varphi \circ \left(\mathrm{id}_\mathbb{R} + \frac{\tau}{K}\right) + \varphi \circ \left(\mathrm{id}_\mathbb{R} - \frac{\tau}{K}\right)\right] \ , \ \frac{K}{2}\,\mathrm{sgn}(\tau)\left(\chi_{[-|\tau|/K,|\tau|/K]} * \varphi\right)$$

for every $\varphi \in C_0^\infty(\mathbb{R}, \mathbb{C})$, where τ runs through the elements of $\mathbb{R}$, from the classical solution formula (5), lead on bounded linear operators in $L^2_\mathbb{C}(\mathbb{R}, \rho)$. For such ρ, according to the linear extension theorem, every member of the family has a unique extension to a bounded linear operator on $L^2(\mathbb{R}, \rho)$, and the resulting family of bounded linear operators on $L^2(\mathbb{R}, \rho)$ can be viewed as resulting from a functional analytic interpretation of (2).

The used methods can be generalized to wave equations on stationary globally hyperbolic space-times with horizons in higher dimensions, such Schwarzschild and Kerr space-times.

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6 Appendix

Assumption 6.1. In the following, we denote for every $n \in \mathbb{N}^*$ by v^n the Lebesgue measure on $\mathbb{R}^n$, $\mathcal{S}_\mathbb{C}(\mathbb{R})$ the space of rapidly decreasing test functions on $\mathbb{R}$, $\mathrm{id}_\mathbb{R}$ the identical function on $\mathbb{R}$, $F_0 : \mathcal{S}_\mathbb{C}(\mathbb{R}) \to \mathcal{S}_\mathbb{C}(\mathbb{R})$ the Fourier transformation, defined by

$$F_0(f)(k) := (2\pi)^{-1/2} \int_\mathbb{R} e^{-ik.\mathrm{id}_\mathbb{R}} f dv^1$$

for every $k \in \mathbb{R}$ and $f \in \mathcal{S}_\mathbb{C}(\mathbb{R})$, $F_1 : L^1_\mathbb{C}(\mathbb{R}) \to C_\infty(\mathbb{R}, \mathbb{C})$ the Fourier transformation defined by

$$F_1(f)(k) := (2\pi)^{-1/2} \int_\mathbb{R} e^{-ik.\mathrm{id}_\mathbb{R}} f dv^1$$

for every $k \in \mathbb{R}$ and $f \in L^1_\mathbb{C}(\mathbb{R})$, and by F_2 the unitary Fourier transformation that is induced by F_0 on $L^2_\mathbb{C}(\mathbb{R})$. Further, for every complex-valued function that is a.e. defined and measurable on $\mathbb{R}$, we denote by T_g the maximal multiplication operator with

g in $L^2_{\mathbb{C}}(\mathbb{R})$. Finally, A is closure of the densely-defined, linear, positive symmetric and essentially self-adjoint operator in $L^2_{\mathbb{C}}(\mathbb{R})$

$$A_0 := \begin{pmatrix} C_0^\infty(\mathbb{R},\mathbb{C}) \to L^2_{\mathbb{C}}(\mathbb{R}) \\ f \mapsto -\frac{1}{K^2}\, f'' \end{pmatrix} \ ,$$

and $\bar{p}_\xi$ is the closure of the densely-defined, linear, symmetric and essentially self-adjoint operator p_ξ in $L^2_{\mathbb{C}}(\mathbb{R})$ given by

$$p_\xi := \begin{pmatrix} C_0^\infty(\mathbb{R},\mathbb{C}) \to L^2_{\mathbb{C}}(\mathbb{R}) \\ f \mapsto \frac{i}{K} f' \end{pmatrix} \ .$$

In particular, the spectrum of $\bar{p}_\xi$ is given by $\mathbb{R}$, and $\bar{p}_\xi$ is the infinitesimal generator of the strongly continuous one-parameter unitary group $U : \mathbb{R} \to L(L^2_{\mathbb{C}}(\mathbb{R}), L^2_{\mathbb{C}}(\mathbb{R}))$ given by

$$U(\tau) := \begin{pmatrix} L^2_{\mathbb{C}}(\mathbb{R}) \to L^2_{\mathbb{C}}(\mathbb{R}) \\ f \mapsto f \circ \left(\mathrm{id}_{\mathbb{R}} - \frac{\tau}{K}\right) \end{pmatrix} \left(= e^{i\tau\bar{p}_\xi}\right)$$

for every $\tau \in \mathbb{R}$.

Theorem 6.2. Then

$$\bar{p}_\xi^2 = A \ ,$$

$\bar{p}_\xi$ commutes with A in the strong sense, and

$$\begin{aligned} \cos(\tau\bar{p}_\xi)g_1 &= \frac{1}{2}\left[g_1 \circ \left(\mathrm{id}_{\mathbb{R}} + \frac{\tau}{K}\right) + g_1 \circ \left(\mathrm{id}_{\mathbb{R}} - \frac{\tau}{K}\right)\right] \ , \\ \frac{\sin(\tau\bar{p}_\xi)}{\bar{p}_\xi}\, g_2 &= \frac{K}{2}\ \mathrm{sgn}(\tau)\left(\chi_{{}_{[-|\tau|/K,|\tau|/K]}} * g_2\right) \end{aligned} \tag{13}$$

for every $g_1, g_2 \in L^2_{\mathbb{C}}(\mathbb{R})$, where $\circ$ denotes composition,

$$\mathrm{sgn} := \chi_{{}_{(0,\infty)}} - \chi_{{}_{(-\infty,0)}} \ ,$$

and $*$ denotes the usual convolution product.

Proof. As a square of a densely-defined, linear and self-adjoint operator, $\bar{p}_\xi^2$ is a densely-defined, linear and self-adjoint operator in $L^2_{\mathbb{C}}(\mathbb{R})$. Obviously, it follows that $\bar{p}_\xi^2 \supset A_0$ and hence, since A_0 is in particular essentially self-adjoint, that $\bar{p}_\xi^2 = \bar{A}_0 = A$.

Further, it follows that

$$\exp(i\tau\bar{p}_\xi)A_0 \subset A_0 \exp(i\tau\bar{p}_\xi) \ .$$

For the proof, we note that

$$\exp(i\tau\bar{p}_\xi)A_0 f = -\frac{1}{K^2}\ \exp(i\tau\bar{p}_\xi)f'' = -\frac{1}{K^2}\left(f'' \circ \left(\mathrm{id}_{\mathbb{R}} - \frac{\tau}{K}\right)\right)$$

$$= -\frac{1}{K^2}\left(f \circ \left(\mathrm{id}_{\mathbb{R}} - \frac{\tau}{K}\right)\right)'' = A_0\left(f \circ \left(\mathrm{id}_{\mathbb{R}} - \frac{\tau}{K}\right)\right) = A_0 \exp(i\tau\bar{p}_\xi) f \ .$$

for $\tau \in \mathbb{R}$ and $f \in C_0^\infty(\mathbb{R}, \mathbb{C})$. Hence it follows also that

$$\exp(i\tau\bar{p}_\xi)A \subset A\exp(i\tau\bar{p}_\xi)$$

which implies that $\bar{p}_\xi$ commutes with A in the strong sense.

For the proof of (13), we note that for $\tau \in [0, \infty)$, $g \in L^2_{\mathbb{C}}(\mathbb{R})$ and $h \in C_0^\infty(\mathbb{R}, \mathbb{C})$

$$\chi_{[-\tau,\tau]} * g \in C_\infty(\mathbb{R}, \mathbb{C}) \cap L^2_{\mathbb{C}}(\mathbb{R}) \tag{14}$$

and that

$$\left(\chi_{[-\tau,\tau]} \circ p_2\right) \cdot (g \circ (p_1 - p_2)) \cdot (h \circ p_1) \tag{15}$$

is v^2-summable, where p_1, p_2 denote the coordinate projections of $\mathbb{R}^2$ onto the first and second coordinate, respectively. For the proof, let $\tau \in [0, \infty)$, $g \in L^2_{\mathbb{C}}(\mathbb{R})$ and $h \in C_0^\infty(\mathbb{R}, \mathbb{C})$. Since $\chi_{[-\tau,\tau]} \in L^2_{\mathbb{C}}(\mathbb{R})$ and

$$F_2\, \chi_{[-\tau,\tau]} = \sqrt{\frac{2}{\pi}}\, \widehat{\frac{\sin(\tau.\mathrm{id}_{\mathbb{R}})}{\mathrm{id}_{\mathbb{R}}}} \ ,$$

where

$$\widehat{\frac{\sin(\tau.\mathrm{id}_{\mathbb{R}})}{\mathrm{id}_{\mathbb{R}}}}$$

denotes the extension of

$$\frac{\sin(\tau.\mathrm{id}_{\mathbb{R}})}{\mathrm{id}_{\mathbb{R}}} \in C(\mathbb{R}^*, \mathbb{C})$$

to an element of $C_\infty(\mathbb{R}, \mathbb{C})$, it follows that

$$\chi_{[-\tau,\tau]} * g = F_1\left(\sqrt{\frac{2}{\pi}}\, \widehat{\frac{\sin(\tau.\mathrm{id}_{\mathbb{R}})}{\mathrm{id}_{\mathbb{R}}}} \cdot [(F_2 g) \circ (-\mathrm{id}_{\mathbb{R}})]\right) \ .$$

Since in particular, the argument of F_1 in the latter equality is also contained in $L^2_{\mathbb{C}}(\mathbb{R})$, (14) follows. Further, if $N \in \mathbb{N}^*$ is such that $\mathrm{supp}(h) \subset [-N, N]$, then

$$\begin{aligned}&\left(\chi_{[-\tau,\tau]} \circ p_2\right) \cdot (g \circ (p_1 - p_2)) \cdot (h \circ p_1) \\ &= \left(\chi_{[-\tau,\tau]} \circ p_2\right) \cdot \left[\left(\chi_{[-(N+\tau),(N+\tau)]} \cdot g\right) \circ (p_1 - p_2)\right] \cdot (h \circ p_1) \ .\end{aligned}$$

Since $\chi_{[-(N+\tau),(N+\tau)]} \in L^2_{\mathbb{C}}(\mathbb{R})$, it follows that $\chi_{[-(N+\tau),(N+\tau)]} \cdot g \in L^1_{\mathbb{C}}(\mathbb{R})$ and hence, since also $\chi_{[-\tau,\tau]} \in L^1_{\mathbb{C}}(\mathbb{R})$, it follows from a known result in connection with convolution products that

$$\left(\chi_{[-\tau,\tau]} \circ p_2\right) \cdot \left[\left(\chi_{[-(N+\tau),(N+\tau)]} \cdot g\right) \circ (p_1 - p_2)\right]$$

is v^2-summable. Finally, since $h \circ p_1$ is bounded and continuous, it follows from integration theory that (15) is v^2-summable.

Finally, we obtain from direct calculation that

$$\cos(\tau \bar{p}_\xi) f = \frac{1}{2} \left[\exp(i\tau\bar{p}_\xi) + \exp(-i\tau\bar{p}_\xi) \right] f = \frac{1}{2} \left[f \circ \left(\mathrm{id}_{\mathbb{R}} - \frac{\tau}{K} \right) + f \circ \left(\mathrm{id}_{\mathbb{R}} + \frac{\tau}{K} \right) \right]$$

as well as

$$\begin{aligned}
&\langle \frac{\sin(\tau\bar{p}_\xi)}{\bar{p}_\xi} g | h \rangle = \mathrm{sgn}(\tau) \, \langle \frac{\sin(|\tau|\bar{p}_\xi)}{\bar{p}_\xi} g | h \rangle = \mathrm{sgn}(\tau) \, \langle g | F_1 \left(\frac{1}{2} \, \chi_{[-|\tau|,|\tau|]} \right) (\bar{p}_\xi) h \rangle \\
&= \frac{1}{2} \, \mathrm{sgn}(\tau) \int_{-|\tau|}^{|\tau|} \langle \exp(it\bar{p}_\xi) g | h \rangle \, dt = \frac{K}{2} \, \mathrm{sgn}(\tau) \int_{-|\tau|/K}^{|\tau|/K} \langle g \circ (\mathrm{id}_{\mathbb{R}} - t) | h \rangle \, dt \\
&= \frac{K}{2} \, \mathrm{sgn}(\tau) \int_{\mathbb{R}^2} \left(\chi_{[-|\tau|/K,|\tau|/K]} \circ p_2 \right) \cdot (g^* \circ (p_1 - p_2)) \cdot (h \circ p_1) dv^2 \\
&= \frac{K}{2} \, \mathrm{sgn}(\tau) \int_{\mathbb{R}} \left(\chi_{[-|\tau|/K,|\tau|/K]} * g \right)^* \cdot h dv^1 = \langle \frac{K}{2} \, \mathrm{sgn}(\tau) \left(\chi_{[-|\tau|/K,|\tau|/K]} * g \right) | h \rangle \ ,
\end{aligned}$$

for $\tau \in \mathbb{R}$, $f, g \in L^2_{\mathbb{C}}(\mathbb{R})$ and $h \in C_0^\infty(\mathbb{R}, \mathbb{C})$. Since $C_0^\infty(\mathbb{R}, \mathbb{C})$ is dense in $L^2_{\mathbb{C}}(\mathbb{R})$, from the latter follows that

$$\frac{\sin(\tau\bar{p}_\xi)}{\bar{p}_\xi} g = \frac{K}{2} \, \mathrm{sgn}(\tau) \chi_{[-|\tau|/K,|\tau|/K]} * g \ .$$

□

Assumption 6.3. In addition, we denote by $B^{\pm 1/2}$ the multiplikation operator by $T_{e^{\mp K\xi}}$ in $L^2_{\mathbb{C}}(\mathbb{R})$. Further, we denote by $U^s_{\mathbb{C}}(\mathbb{R})$ the set of bounded complex-valued functions on $\mathbb{R}$ with component functions that are strongly measurable, in the sense that they are everywhere on $\mathbb{R}$ the limit of a sequence of step functions. Finally, $\mathcal{A}$ denotes the set of all complex-valued functions f on $\mathbb{R} \times [-1, 0]$ satisfying

1. $f(\cdot, 0), f(\cdot, 1) \in U^s_{\mathbb{C}}(\mathbb{R})$,

2. a.e. on $\mathbb{R}$:

$$\lim_{y \to 0-} f(\cdot, y) = f(\cdot, 0) \ , \ \lim_{y \to (-1)+} f(\cdot, y) = f(\cdot, -1) \ ,$$

3. $f|_{\mathbb{R} \times (-1,0)}$ is holomorphic, and there are $C \geq 0$, $N \in \mathbb{N}$ such that

$$|f(z)| \leq C(1 + |z|)^N$$

for every $z \in \mathbb{R} \times [-1, 0]$.

Theorem 6.4. For every $f \in \mathcal{A}$

$$C_0^\infty(\mathbb{R},\mathbb{C}) \subset D(B^{-1/2} f(\bar{p}_\xi, 0) B^{1/2}) \text{ and } \overline{B^{-1/2} f(\bar{p}_\xi, 0) B^{1/2}} = f(\bar{p}_\xi, -1) \ ,$$

where the overline on top of the expression containing $f(\bar{p}_\xi, 0)$ indicates closure in the operator norm of $L(L^2_{\mathbb{C}}(\mathbb{R}), L^2_{\mathbb{C}}(\mathbb{R}))$.

Proof. In a first step, we prove an auxiliary result. For this purpose, let $f \in \mathcal{A}$, $\varphi \in C(\mathbb{R} \times [-1,0], \mathbb{C})$ such that $\varphi|_{\mathbb{R}\times(-1,0)}$ is holomorphic and $(\mathrm{id}_{\mathbb{C}})^k \cdot \varphi$ is bounded for every $k \in \mathbb{N}$. Then,

$$\int_{\mathbb{R}} f(\cdot, 0) \cdot \varphi(\cdot, 0)\, dv^1 = \int_{\mathbb{R}} f(\cdot, 1) \cdot \varphi(\cdot, 1)\, dv^1 \ .$$

The proof is a straighforward application of Cauchy's integral theorem for rectangular paths and Lebesgue's dominated convergence theorem.

Further, we note that from a well-known theorem of Paley-Wiener, see e.g., [12] Vol. I, Sect IX.3, follows that for any $g \in C_0^\infty(\mathbb{R},\mathbb{C})$ the corresponding function

$$\begin{pmatrix} \mathbb{R} \times [-1,0] \to \mathbb{C} \\ z \mapsto \int_{\mathbb{R}} e^{-iz.\mathrm{id}_{\mathbb{R}}} \cdot g\, dv^1 \end{pmatrix}$$

has the properties that were required for φ in the previous auxiliary result.

As a further auxiliary result, we note that

$$\begin{pmatrix} \mathbb{R} \to \mathbb{C} \\ \tau \mapsto \langle h_1 | \exp(i\tau\bar{p}_\xi) h_2 \rangle \end{pmatrix} \in C_0^\infty(\mathbb{R},\mathbb{C})$$

for every $h_1, h_2 \in C_0^\infty(\mathbb{R},\mathbb{C})$. The latter follows from the identities

$$\begin{aligned}
&\langle h_1 | \exp(i\tau\bar{p}_\xi) h_2 \rangle = \langle h_1 | h_2 \circ \left(\mathrm{id}_{\mathbb{R}} - \frac{\tau}{K}\right) \rangle = \int_{\mathbb{R}} h_1^* \left[h_2 \circ \left(\mathrm{id}_{\mathbb{R}} - \frac{\tau}{K}\right)\right] dv^1 \\
&= \frac{1}{K} \int_{\mathbb{R}} \left(h_1^* \circ \left(K^{-1}.\mathrm{id}_{\mathbb{R}}\right)\right) \left[h_2 \circ \left(-K^{-1}.\mathrm{id}_{\mathbb{R}}\right) \circ (\tau - \mathrm{id}_{\mathbb{R}})\right] dv^1 \\
&= \frac{1}{K} \left[\left(h_1^* \circ \left(K^{-1}.\mathrm{id}_{\mathbb{R}}\right)\right) * \left(h_2 \circ \left(-K^{-1}.\mathrm{id}_{\mathbb{R}}\right)\right)\right](\tau) \\
&= \frac{\sqrt{2\pi}}{K} F_0 \left[F_0^{-1} \left(h_1^* \circ \left(K^{-1}.\mathrm{id}_{\mathbb{R}}\right)\right) \cdot F_0^{-1} \left(h_2 \circ \left(-K^{-1}.\mathrm{id}_{\mathbb{R}}\right)\right)\right](\tau)
\end{aligned}$$

which show that

$$\begin{pmatrix} \mathbb{R} \to \mathbb{C} \\ \tau \mapsto \langle h_1 | \exp(i\tau\bar{p}_\xi) h_2 \rangle \end{pmatrix} \in C_0(\mathbb{R},\mathbb{C}) \cap \mathcal{S}_{\mathbb{C}}(\mathbb{R}) = C_0^\infty(\mathbb{R},\mathbb{C}) \ .$$

As a final auxilary result, we note that $C_0^\infty(\mathbb{R},\mathbb{C})$ is a core for $B^{-1/2}$. The latter follows from the facts that $C_0^\infty(\mathbb{R},\mathbb{C})$ is dense in $L^2_{\mathbb{C}}(\mathbb{R})$, contained in $D(B^{-1/2})$ and invariant under the strongly continuous one-parameter unitary group

$$\begin{pmatrix} \mathbb{R} \to L(L^2_{\mathbb{C}}(\mathbb{R}), L^2_{\mathbb{C}}(\mathbb{R})) \\ t \mapsto T_{\exp(it.\exp(K.\mathrm{id}_{\mathbb{R}}))} \end{pmatrix}$$

that is generated by $B^{-1/2}$, where for every $t \in \mathbb{R}$, the corresponding $T_{\exp(it.\exp(K.\mathrm{id}_{\mathbb{R}}))}$ denotes the maximal multiplication operator in $L^2_{\mathbb{C}}(\mathbb{R})$ by the function $\exp(it.\exp(K.\mathrm{id}_{\mathbb{R}}))$.

For the proof of our main result, let $f \in \mathcal{A}$. Since $B^{-1/2}$ is in particular self-adjoint and $C_0^\infty(\mathbb{R},\mathbb{C})$ is a core for $B^{-1/2}$, it follows for $h_1 \in C_0^\infty(\mathbb{R},\mathbb{C})$ that

$$f(\bar{p}_\xi, 0)B^{1/2}h_1 \in D(B^{-1/2})$$

if and only if the linear form

$$\langle f(\bar{p}_\xi, 0)B^{1/2}h_1 | B^{-1/2}\cdot\rangle\,|_{C_0^\infty(\mathbb{R},\mathbb{C})}$$

is bounded. With the help of the previous auxiliar results, we conclude for $h_1, h_2 \in C_0^\infty(\mathbb{R},\mathbb{C})$ that

$$\begin{aligned}
&\left(\langle f(\bar{p}_\xi, 0)B^{1/2}h_1 | B^{-1/2}h_2\rangle\right)^* \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} f(\cdot, 0)F_1 \begin{pmatrix} \mathbb{R} \to \mathbb{C} \\ \tau \mapsto \langle B^{-1/2}h_2 | \exp(i\tau\bar{p}_\xi)B^{1/2}h_1\rangle \end{pmatrix} dv^1 \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} f(\cdot, 0)F_1 e^{\mathrm{id}_{\mathbb{R}}} \begin{pmatrix} \mathbb{R} \to \mathbb{C} \\ \tau \mapsto \langle h_2 | \exp(i\tau\bar{p}_\xi)h_1\rangle \end{pmatrix} dv^1 \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} f(\cdot, -1)F_1 \begin{pmatrix} \mathbb{R} \to \mathbb{C} \\ \tau \mapsto \langle h_2 | \exp(i\tau\bar{p}_\xi)h_1\rangle \end{pmatrix} dv^1 \\
&= \langle h_2 | f(\bar{p}_\xi, -1)h_1\rangle = (\langle f(\bar{p}_\xi, -1)h_1 | h_2\rangle)^* \ .
\end{aligned}$$

From the latter, we conclude from the self-adjointness of $B^{-1/2}$ that for $h_1 \in C_0^\infty(\mathbb{R},\mathbb{C})$ that

$$f(\bar{p}_\xi, 0)B^{1/2}h_1 \in D(B^{-1/2})$$

and that

$$B^{-1/2}f(\bar{p}_\xi, 0)B^{1/2}h_1 = f(\bar{p}_\xi, -1)h_1 \ .$$

Finally, since $C_0^\infty(\mathbb{R},\mathbb{C})$ is dense in $L^2_{\mathbb{C}}(\mathbb{R})$, the latter shows that $B^{-1/2}f(\bar{p}_\xi, 0)B^{1/2}$ is a densely-defined, linear and bounded operator in $L^2_{\mathbb{C}}(\mathbb{R})$, whose extension to a bounded linear operator on $L^2_{\mathbb{C}}(\mathbb{R})$ is given by $f(\bar{p}_\xi, -1)$. $\square$

Theorem 6.5.

$$V \circ \overline{B^{-1/2} \cos(\tau \bar{p}_\xi) B^{1/2}}\, V^{-1} f = \frac{1}{2} \left[f \circ \left(\mathrm{id}_{\mathbb{R}} + \frac{\tau}{K} \right) + f \circ \left(\mathrm{id}_{\mathbb{R}} - \frac{\tau}{K} \right) \right] \ ,$$
$$V \circ \overline{B^{-1/2} \frac{\sin(\tau \bar{p}_\xi)}{\bar{p}_\xi} B^{1/2}}\, V^{-1} g = \frac{K}{2} \ \mathrm{sgn}(\tau) \left(\chi_{[-|\tau|/K, |\tau|/K]} * g \right) \tag{16}$$

for all $f, g \in L^2_{\mathbb{C}}(\mathbb{R}, e^{2K\xi})$, where the overline on top of the expressions starting with $B^{-1/2}$ indicates closure in the operator norm of $L(L^2_{\mathbb{C}}(\mathbb{R}), L^2_{\mathbb{C}}(\mathbb{R}))$.

Proof. In a first step, we note that

$$\cos(\tau.\mathrm{id}_{\mathbb{R}}) \ , \ \frac{\widehat{\sin(\tau.\mathrm{id}_{\mathbb{R}})}}{\mathrm{id}_{\mathbb{R}}} \in \mathcal{A} \ .$$

Hence according to Theorem 6.4, the linear operators

$$B^{-1/2} \cos(\tau \bar{p}_\xi) B^{1/2} \ , \ B^{-1/2} \frac{\sin(\tau \bar{p}_\xi)}{\bar{p}_\xi} B^{1/2}$$

are in particular densely-defined, with domains containing $C^\infty_0(\mathbb{R}, \mathbb{C})$, and bounded. As a consequence, these operators have unique extensions to bounded linear operators on $L^2_{\mathbb{C}}(\mathbb{R})$.

Further,

$$B^{1/2} V^{-1} \varphi = e^{-K\xi} e^{K\xi} \varphi = \varphi \ , \ V B^{-1/2} f = e^{-K\xi} e^{K\xi} f = f$$

for every $\varphi \in C^\infty_0(\mathbb{R}, \mathbb{C})$ and $f \in D(B^{-1/2})$. Hence it follows from Theorem 6.2 that

$$V \circ B^{-1/2} \cos(\tau \bar{p}_\xi) B^{1/2} V^{-1} \varphi = \frac{1}{2} \left[\varphi \circ \left(\mathrm{id}_{\mathbb{R}} + \frac{\tau}{K} \right) + \varphi \circ \left(\mathrm{id}_{\mathbb{R}} - \frac{\tau}{K} \right) \right] \ ,$$
$$V \circ B^{-1/2} \frac{\sin(\tau \bar{p}_\xi)}{\bar{p}_\xi} B^{1/2} V^{-1} \varphi = \frac{K}{2} \ \mathrm{sgn}(\tau) \left(\chi_{[-|\tau|/K, |\tau|/K]} * \varphi \right)$$

for every $\varphi \in C^\infty_0(\mathbb{R}, \mathbb{C})$.

Finally, for $\tau \in \mathbb{R}$, by

$$f \circ \left(\mathrm{id}_{\mathbb{R}} \pm \frac{\tau}{K} \right) \ , \ \chi_{[-|\tau|/K, |\tau|/K]} * f$$

for every $f \in L^2_{\mathbb{C}}(\mathbb{R}, e^{2K\xi})$, there are defined bounded linear operators on $L^2_{\mathbb{C}}(\mathbb{R}, e^{2K\xi})$.

For the proof, we note that

$$e^{\mp\tau} . V \circ e^{i(\mp\tau)\bar{p}_\xi} \circ V^{-1} f = f \circ \left(\mathrm{id}_{\mathbb{R}} \pm \frac{\tau}{K} \right)$$

for every $f \in L^2_{\mathbb{C}}(\mathbb{R}, e^{2K\xi})$. Also, by

$$\begin{pmatrix} L^2_{\mathbb{C}}(\mathbb{R}) \to L^2_{\mathbb{C}}(\mathbb{R}) \\ f \mapsto (\exp(K.\mathrm{id}_{\mathbb{R}}) \cdot \chi_{[-|\tau|/K,|\tau|/K]}) * f \end{pmatrix}$$

there is defined a bounded linear operator on $L^2_{\mathbb{C}}(\mathbb{R})$. The latter can be seen as follows. Since $\exp(K.\mathrm{id}_{\mathbb{R}}) \cdot \chi_{[-|\tau|/K,|\tau|/K]} \in L^2_{\mathbb{C}}(\mathbb{R})$,

$$\big(\exp(K.\mathrm{id}_{\mathbb{R}}) \cdot \chi_{[-|\tau|/K,|\tau|/K]})\big) * f \in C_\infty(\mathbb{R}, \mathbb{C}) \cap L^2_{\mathbb{C}}(\mathbb{R})$$

Further,

$$\| \big(\exp(K.\mathrm{id}_{\mathbb{R}}) \cdot \chi_{[-|\tau|/K,|\tau|/K]})\big) * f\|_2 \leq \frac{2|\tau|}{K}\, e^{|\tau|} \|f\|_2$$

for every $f \in L^2_{\mathbb{C}}(\mathbb{R})$. Finally, we note that

$$\begin{aligned} & V \circ \big(\exp(K.\mathrm{id}_{\mathbb{R}}) \cdot \chi_{[-|\tau|/K,|\tau|/K]})\big) * V^{-1} g \\ & = \exp(-K.\mathrm{id}_{\mathbb{R}}) \left[\big(\exp(K.\mathrm{id}_{\mathbb{R}}) \cdot \chi_{[-|\tau|/K,|\tau|/K]})\big) * (\exp(K.\mathrm{id}_{\mathbb{R}}) \cdot g)\right] \\ & = \chi_{[-|\tau|/K,|\tau|/K]} * g \end{aligned}$$

for every $g \in L^2_{\mathbb{C}}(\mathbb{R}, e^{2K\xi})$, and collecting the obtained information we arrive at (16).

□

References

[1] Beyer, H R 1991, *Zur kanonischen lokalen Quantisierung des freien neutralen Skalarfeldes auf einer statischen Raum-Zeit (On the canonical local quantization of the free scalar field on a static spacetime)*, Ph.D. Thesis, University of Hamburg: Germany.

[2] Beyer H R 1991, *Remarks on Fulling's quantization*, Class. Quantum Grav., **8**, 1091-1112.

[3] Beyer H. R. 1999, *On the completeness of the quasinormal modes of the Pöschl-Teller potential*, Commun. Math. Phys. , **204**, 397-423.

[4] Beyer H R 2001, *On the stability of the Kerr metric*, Commun. Math. Phys., **221**, 659-676.

[5] Beyer H R 2011, *On the stability of the massive scalar field in Kerr space-time*, J. Math. Phys., **52**, 102502,1-21.

[6] Beyer H R 2007, *Beyond partial differential equations: A course on linear and quasi-linear abstract hyperbolic evolution equations*, LNM 1898, Springer: Berlin.

[7] Fulling S A 1973, *Nonuniqueness of Canonical Field Quantization in Riemannian Space-Time*, Phys. Rev. D, **7**, 2850-2862.

[8] Fulling S A 1989, *Aspects of quantum field theory in curved spacetime*, Cambridge University Press, Cambridge.

[9] Horowitz G T, Hubeny V E 2000, *Quasinormal Modes of AdS Black Holes and the Approach to Thermal Equilibrium*, Phys. Rev. D, **62**, 024027, 1-11.

[10] Kay B S 1985 *The double-wedge algebra for quantum fields on Schwarschild and Minkowski spacetimes* Commun. Math. Phys., **100**, 57-81.

[11] Kay B S and Wald R M 1987 *Linear stability of Schwarzschild under perturbations which are non-vanishing on the bifurcation 2-sphere*, Class. Quantum Grav.,**4**, 893-898.

[12] Reed M and Simon B 1980, 1975, 1979, 1978 *Methods of Mathematical Physics Volume I, II, III, IV*, Academic: New York.

[13] Soper D E 2008, *Classical field theory*, Dover: New York.

[14] Wald R M 1994, *Quantum field theory in curved spacetime and black hole thermodynamics*, University of Chicago Press: Chicago.